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The Exhaustion Numbers of the Generalized Quaternion Groups

Chen, H. V.* and Sin, C. S.

Department of Mathematical and Actuarial Sciences, Lee Kong Chian Faculty of Engineering and Science, Universiti Tunku Abdul Rahman, Selangor, Malaysia

E-mail: chenhv@utar.edu.my
**Corresponding author*

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Abstract

Let G be a finite group and let T be a non-empty subset of G. For any positive integer k, let $T^k = \{t_1 \dots t_k \mid t_1, \dots, t_k \in T\}$. The set T is called exhaustive if $T^n = G$ for some positive integer n where the smallest positive integer n, if it exists, such that $T^n = G$ is called the exhaustion number of T and is denoted by $e(T)$. If $T^k \neq G$ for any positive integer k, then T is a non-exhaustive subset and we write $e(T) = \infty$. In this paper, we investigate the exhaustion numbers of subsets of the generalized quaternion group $Q_{2^n} = \langle x, y \mid x^{2^{n-1}} = 1, x^{2^{n-2}} = y^2, yx = x^{2^{n-1}-1}y\rangle$ where $n \geq 3$. We show that Q_{2^n} has no exhaustive subsets of size 2 and that the smallest positive integer k such that any subset $T \subseteq Q_{2^n}$ of size greater than or equal to k is exhaustive is $2^{n-1}+1$. We also show that for any integer $k \in \{3, \dots, 2^n\}$, there exists an exhaustive subset T of Q_{2^n} such that $|T| = k$.

Keywords: finite group; generalized quaternion group; exhaustion number.

1 Introduction

Factoring a finite abelian group into subsets was first initiated by Hajós [8] to solve a famous geometry problem by H. Minkowski in 1941. In the past few decades, study of group factorization has received numerous research attentions (see [9, 12]) and they found its applications in various field such as geometry of tiling, code theory, cryptography, graph theory and etc (see [5, 6]). The current research has renew focused on factoring nonabelian groups into subsets (see [1, 11]). The study by Sahai and Ansari [13] adds to the growing body of research in this area and provides insights into the properties of non-abelian groups and how they can be factored.

The generalized quaternion groups are nonabelian groups, which have been widely used in various applications. Every abelian subgroup of generalized quaternion groups is cyclic. The algebraic structure of the generalized quaternion groups has attracted the interest of researchers from various background in representation theory [10] and information theory [14]. Linear complementary dual (LCD) codes which are a class of linear codes, have been extensively studied by many researchers recently. In [7], the generalized quaternion algebras is focused to construct linear complementary dual (LCD) codes and self-orthogonal codes.

Let G be a finite group and let T be a non-empty subset of G. For any positive integer k, let $T^k = \{t_1 \dots t_k \mid t_1, \dots, t_k \in T\}$. The set T is called exhaustive if $T^n = G$ for some positive integer n. The smallest positive integer n, if it exists, such that $T^n = G$ is called the exhaustion number of T and is denoted by $e(T)$. If $T^k \neq G$ for any positive integer k, then T is a non-exhaustive subset and we write $e(T) = \infty$.

Early work on exhaustion numbers have focused mainly on subsets of finite abelian groups. In [3], the authors studied some properties of exhaustive and non-exhaustive subsets of finite groups. A characterization of non-exhaustive subsets of finite groups is found and it can be used to obtain an upper bound for the size of a non-exhaustive subset. In [4], Chin completely determined the exhaustion numbers of subsets of abelian groups that are in arithmetic progression as well as the exhaustion numbers of various other subsets of abelian groups. Recent work on exhaustion numbers have extended to subsets of finite non-abelian groups, in particular, the dihedral groups.

The exhaustive subsets of size 2 are fundamental to the constructions of the exhaustive subsets of size m for positive integer $m > 2$. In [2], the authors studied the exhaustion numbers of all the $n(2n-1)$ subsets of the size 2 of dihedral group $D_{2n} = \langle x, y | x^n = y^2 = 1, xy = yx^{n-1}\rangle$ of order $2n$ for $n \geq 3$. The existence of exhaustion 2-subsets for all the subsets $B = \{h_1, h_2\}$ in the dihedral group D_{2p} is proved using the notion of group ring, where p is an odd prime number, $h_1 \in \langle x\rangle$ and $h_2 \in \langle x\rangle y$ in [15].

Let G be a finite group. For any group element $g \in G$ and non-empty subset S of the group G, the notation $\langle g\rangle$ denotes the cyclic group generated by g whereas $gS = \{gs \mid s \in S\}$. Similarly, $Sg = \{sg \mid s \in S\}$. In this paper, we study the products of k subsets of generalized quaternion groups Q_{2^n} and investigate the exhaustion numbers of subsets of Q_{2^n} for $n \geq 3$. Throughout this paper, let $Q_{2^n} = \langle x, y \mid x^{2^{n-1}} = 1, x^{2^{n-2}} = y^2, yx = x^{2^{n-1}-1}y\rangle$, the generalized quaternion group of order 2^n where $n \geq 3$. Thus, $Q_{2^n} = \langle x\rangle \cup \langle x\rangle y$. We first identify the largest non-exhaustive subsets of Q_{2^n} in Section 2 and show that Q_{2^n} does not have any exhaustive subsets of size 2. This is followed in Section 3 with a study of the exhaustion numbers of various subsets of Q_{2^n}, $n \geq 3$. In particular, we show that smallest positive integer k such that any subset $T \subseteq Q_{2^n}$ of size greater than or equal to k is exhaustive is $2^{n-1} + 1$ for $n \geq 3$. We also show that for any integer $k \in \{3, \dots, 2^n\}$, there exists an exhaustive subset T of Q_{2^n} such that $|T| = k$.

2 Non-Exhaustive Subsets of Q_{2^n}, $n \geq 3$

In this section, we identify some non-exhaustive subsets of the generalized quaternion groups Q_{2^n}. It is clear that any proper subgroup H of a finite group G forms a non-exhaustive subset of the group since $H^n = H \subsetneq G$ for any positive integer n. Thus, all subgroups of Q_{2^n}, $n \geq 3$ are non-exhaustive.

We begin by stating two results that hold for all finite groups – the first result tells us that subsets of exhaustive subsets are exhaustive whereas the second result tells us that subsets of non-exhaustive subsets are non-exhaustive.

Proposition 2.1. *Let G be a finite group and let T, S be non-empty subsets of G. If $T \subseteq S$ and $e(T)$ exists, then $e(S) \leq e(T)$.*

Proof. Suppose that $e(T)$ exists and $e(T) = k$ for some positive integer k. So $T^k = G$, and it follows that $S^k = G$ since $T \subseteq S$. Hence, it is clear that $e(S) \leq e(T)$. □

Corollary 2.1. *Let G be a finite group and let T, S be non-empty subsets of G. If $T \subseteq S$ and $e(S) = \infty$, then $e(T) = \infty$.*

Proof. Suppose that $e(T) = k$ for some positive integer k. Since $T \subseteq S$, it follows by Proposition 2.1 that $e(S) \leq e(T)$ which contradicts the fact that $e(S) = \infty$. □

We next list the largest non-exhaustive subsets of Q_{2^n}, $n \geq 3$.

Proposition 2.2. *Let $T_1, \dots, T_5$ be subsets of $Q_{2^n} = \langle x, y \mid x^{2^{n-1}} = 1, x^{2^{n-2}} = y^2, yx = x^{2^{n-1}-1}y\rangle$ $(n \geq 3)$ as follows:*

(i) $T_1 = \{y, xy, x^2y, \dots, x^{2^{n-1}-1}y\}$;

(ii) $T_2 = \{1, x^2, x^4, \dots, x^{2k}, \dots, x^{2^{n-1}-2}, y, x^2y, x^4y, \dots, x^{2k}y, \dots, x^{2^{n-1}-2}y\}$;

(iii) $T_3 = \{1, x^2, x^4, \dots, x^{2k}, \dots, x^{2^{n-1}-2}, xy, x^3y, \dots, x^{2k+1}y, \dots, x^{2^{n-1}-1}y\}$;

(iv) $T_4 = \{x, x^3, \dots, x^{2k+1}, \dots, x^{2^{n-1}-1}, xy, x^3y, \dots, x^{2k+1}y, \dots, x^{2^{n-1}-1}y\}$;

(v) $T_5 = \{x, x^3, \dots, x^{2k+1}, \dots, x^{2^{n-1}-1}, y, x^2y, x^4y, \dots, x^{2k}y, \dots, x^{2^{n-1}-2}y\}$.

Then $|T_i| = 2^{n-1}$ and $e(T_i) = \infty$ for $i = 1, \dots, 5$.

Proof. (i) Since $x^{2^{n-2}} = y^2$ and $yx = x^{2^{n-1}-1}$, we see that $T_1^2 = \{1, x, x^2, \dots, x^{2^{n-1}-1}\}$. Note that T_1^2 is a subgroup of Q_{2^n} and hence, $T_1^{2k} \neq Q_{2^n}$ for any integer k which implies that $e(T_1) = \infty$.

(ii) – (iii) Since $x^{2i+1} \notin T_2^k, T_3^k$ for all $i, k \in \mathbb{N}$, it follows that $e(T_2) = e(T_3) = \infty$.

(iv) – (v) We observe that $T_4^{2k}, T_5^{2k} \not\supseteq \{x, x^3, \dots, x^{2^{n-1}-1}\} \cup \{xy, x^3y, \dots, x^{2^{n-1}-1}y\}$ and T_4^{2k+1}, $T_5^{2k+1} \not\supseteq \{1, x^2, \dots, x^{2^{n-1}-2}\} \cup \{y, x^2y, \dots, x^{2^{n-1}-2}y\}$ for $k \in \mathbb{N}$. Hence, $e(T_4) = e(T_5) = \infty$.

□

We remark that there is no non-exhaustive subsets of Q_{2n} of size greater than 2^{n-1}.

Corollary 2.2. *There exists $T \subseteq Q_{2^n}$ $(n \geq 3)$ where $|T| \in \{1, \dots, 2^{n-1}\}$ such that $e(T) = \infty$.*

Proof. The result is obvious since any subset $S \subseteq T_i$ from Proposition 2.2 is non-exhaustive for all $i = 1, 2, \dots, 5$. □

The following theorem tells us that all subsets of size 2 in Q_{2^n} $(n \geq 3)$ are non-exhaustive.

Theorem 2.1. *Let $T \subseteq Q_{2^n}$, $n \geq 3$. If $|T| = 2$, then $e(T) = \infty$.*

Proof. Let $a, b \in \{0, 1, \dots, 2^{n-1} - 1\}$, $a \neq b$. Clearly, a subset of size 2 in Q_{2^n} may take one of the following forms: $S_1 = \{x^a, x^b\}$, $S_2 = \{x^a y, x^b y\}$, or $S_3 = \{x^a, x^b y\}$. Let $T_1, \dots, T_5$ be the subsets of Q_{2^n} as described in Proposition 2.2.

Note that $S_1 = \{x^a, x^b\} \subset \langle x \rangle$. Since $\langle x \rangle$ is non-exhaustive, it follows by Corollary 2.1 that S_1 is also non-exhaustive. Clearly, $S_2 \subset T_1$ and S_3 is a subset of one of the sets T_2, T_3, T_4 or T_5. Therefore, by Corollary 2.1, $e(S_i) = \infty$ for $i = 2, 3$. Hence, we conclude that $e(S) = \infty$ for any subset $S \subset Q_{2^n}$ of size 2. □

Remark 2.1. By Theorem 2.1 we have that the minimal generating set $\{x, y\}$ of Q_{2^n} is not exhaustive. Thus, a generating set is not necessarily exhaustive although the converse is always true.

3 Exhaustive Subsets of $Q_{2^n}, n \geq 3$

Let G be a finite group and let $T \subseteq G$. If $|T| = 1$, then $|T^m| = 1$ for every positive integer m and hence, $e(T) = \infty$. On the other hand, if $|T| = 2^n$, then $T = Q_{2^n}$ and hence, $e(T) = 1$. Therefore, the extreme cases where $|T| = 1$ and $|T| = 2^n$ are trivial.

We have seen in the previous section that Q_{2^n} does not have any exhaustive subsets of size 2. In this section, we investigate the exhaustion numbers of subsets T of Q_{2^n} for $n \geq 3$, where $|T| \in \{3, \dots, 2^n - 1\}$. By using the group relations in Q_{2^n}, we note that $\langle x \rangle y x^i = \langle x \rangle y$ and $\langle x \rangle y x^i y = \langle x \rangle$ for $i \in \{0, 1, \dots, 2^{n-1} - 1\}$.

By Proposition 2.2 we know that there are non-exhaustive subsets of Q_{2^n} of size 2^{n-1}. The following proposition tells us that any subset T of Q_{2^n} such that $|T| \geq 2^{n-1} + 1$ is exhaustive.

Proposition 3.1. *Let $T \subseteq Q_{2^n}$, $n \geq 3$. If $2^{n-1} + 1 \leq |T| \leq 2^n - 1$, then the exhaustion number, $e(T) = 2$.*

Proof. We begin by assuming that $T \supseteq \langle x \rangle = \{1, x, x^2, \dots, x^{2^{n-1}-1}\}$. Since $|\langle x \rangle| = 2^{n-1}$ and $|T| \geq 2^{n-1} + 1$, so there exists $x^i y \in T$ for some $i \in \{0, 1, \dots, 2^{n-1} - 1\}$. Clearly, $\langle x \rangle y = \langle x \rangle x^i y \subseteq T^2$ and $\langle x \rangle = \langle x \rangle \langle x \rangle \subseteq T^2$. Hence, $e(T) = 2$.

Next, we assume $T \supseteq \langle x \rangle y = \{y, xy, \dots, x^{2^{n-1}-1} y\}$. Since $|\langle x \rangle y| = 2^{n-1}$ and $|T| \geq 2^{n-1} + 1$, so there exists an $x^j \in T$ for some $j \in \{0, 1, \dots, 2^{n-1} - 1\}$. Thus, we have $\langle x \rangle y = \langle x \rangle y x^j, \langle x \rangle = \langle x \rangle y \langle x \rangle y \subseteq T^2$ and hence, $e(T) = 2$.

Now suppose that $T \not\supseteq \langle x\rangle$ and $T \not\supseteq \langle x\rangle y$. Let $E = T \cap \{1, x, \ldots, x^{2^{n-1}-1}\}$ and $F = T \cap \{y, xy, \ldots, x^{2^{n-1}-1}y\}$. Clearly, $E \neq \emptyset$, $F \neq \emptyset$, $E \cap F = \emptyset$ and $T = E \cup F$. We first show that $x^i y \in T^2$ for $i \in \{0, 1, \ldots, 2^{n-1}-1\}$. Suppose that $|E| = 2^{n-1} - j$, where $j \in \{1, \ldots, 2^{n-1}-1\}$. Thus, $|F| = |T| - |E| \geq (2^{n-1}+1) - (2^{n-1}-j) = j+1$. Let $F' = \{x^{\alpha_1}y, \ldots, x^{\alpha_{j+1}}y\} \subseteq F$. Then $E(x^{\alpha_i}y) \neq E(x^{\alpha_k}y)$ and $|E(x^{\alpha_i}y)| = |E|$ for $i \neq k, i, k \in \{1, \ldots, j+1\}$. Hence, $EF' = \{x^k(x^{\alpha_l}y) \mid x^k \in E, x^{\alpha_l}y \in F'\}$ has at least $|E| + j = 2^{n-1}$ elements. Since there are only 2^{n-1} elements of the form $x^i y$ in Q_{2^n}, it follows that $x^i y \in T^2$ for all $i \in \{0, 1, \ldots, 2^{n-1}-1\}$.

Next, we show that $x^i \in T^2$ for all $i \in \{0, 1, \ldots, 2^{n-1}-1\}$. We begin by assuming that $|E| > 2^{n-2}$. Suppose that $x^j \notin T^2$ for some $j \in \{0, 1, \ldots, 2^{n-1}-1\}$. Note that $x^j = 1(x^j) = (x)(x^{j-1}) = (x^2)(x^{j-2}) = \cdots = (x^{2^{n-1}-1})(x^{j-(2^{n-1}-1)})$. Since the powers of x commute with one another, so the number of ways to represent x^j as a product of two elements of $\{1, x, \ldots, x^{2^{n-1}-1}\}$ is 2^{n-2} when j is odd and $2^{n-2}+1$ when j is even. Then since $x^j \notin T^2$, we have $|E| \leq 2^{n-2}$, which is a contradiction. Thus, $x^j \in T^2$ for all $j \in \{0, 1, \ldots, 2^{n-1}-1\}$. This leaves us with the case $|E| \leq 2^{n-2}$, that is, $|F| \geq 2^{n-1} + 1 - 2^{n-2} = 2^{n-2} + 1$.

Note that $x^i = y(x^{2^{n-2}-i}y) = (xy)(x^{2^{n-2}+1-i}) = \cdots = (x^{2^{n-1}-1}y)(x^{2^{n-2}+(2^{n-1}-1)-i}y)$ for any $i \in \{0, 1, \ldots, 2^{n-1}-1\}$. Moreover, $(x^i y)(x^j y) = (x^j y)(x^i y)$ if and only if $i - j \equiv 0 \pmod{2^{n-2}}$. Since $|F| \geq 2^{n-2}+1$, we see that there is at least one pair of distinct elements of F, say $(x^r y, x^s y)$ such that $r - s \equiv 0 \pmod{2^{n-2}}$. That is, the elements $x^r y$ and $x^s y$ commute with one another but do not commute with any other element in F. Then by considering the products $(x^r y)(x^k y)$ for all $x^k y \in F$ and noting that $|F| \geq 2^{n-2}+1$, we see that $x^j \in T^2$ for all $j \in \{0, 1, \ldots, 2^{n-1}-1\}$. Thus, we conclude that $e(T) = 2$. □

Theorem 3.1. *Let $T \subseteq Q_{2^n}$, $n \geq 3$ with $|T| = s$. The smallest positive integer s such that any subset T is exhaustive is $2^{n-1}+1$.*

Proof. From Proposition 3.1, we see that if $2^{n-1}+1 \leq |T| \leq 2^n - 1$, then $e(T) = 2$. The result is obvious since there are non-exhaustive subsets of Q_{2n} of size 2^{n-1} from Proposition 2.2. □

Remark 3.1. Note that the exhaustion numbers of subsets of the same size in a group are not necessarily the same. For example, take $T_1 = \{y, xy, x^2y, \ldots, x^{2^{n-1}-1}y\}$, $T_2 = \{1, x, \ldots, x^{2^{n-1}-2}, y\}$ and $T_3 = \{1, x, \ldots, x^{2^{n-2}-1}, x^{2^{n-2}+1}, x^{2^{n-2}+2}, \ldots, x^{2^{n-1}-1}, y\} \subseteq Q_{2^n}$. Then $|T_1| = |T_2| = |T_3| = 2^{n-1}$ but $e(T_1) = \infty$ (by Proposition 2.2) whereas $e(T_2) = 2$ and $e(T_3) = 3$.

Other than the subsets with exhaustion number 2 given in Proposition 3.1, there also exist subsets of smaller size with exhaustion number 2 as shown in the following proposition.

Proposition 3.2. *Let $n \geq 3$ and $m \in \{2^{n-2}, 2^{n-2}+1, \ldots, 2^{n-1}-2\}$. If $T = \{1, x, x^2, \ldots, x^m, y\} \subseteq Q_{2^n}$, then $e(T) = 2$.*

Proof. Let $T = T_1 \cup \{y\}$, where $T_1 = \{1, x, \ldots, x^m\}$. Note that $1, x, \ldots, x^{2m} \in T_1^2$ and $T_1^2 = \langle x\rangle$ as $m \geq 2^{n-2}$. We see that $T_1 y = \{y, xy, \ldots, x^m y\}$ and $yT_1 = \{y, x^{2^{n-1}-1}y, x^{2^{n-1}-2}y, \ldots, x^{2^{n-1}-m}y\}$.

Since $m \geq 2^{n-2} > 2^{n-2} - \frac{1}{2} = \frac{2^{n-1}-1}{2}$, so $2^{n-1} - 1 < 2m$; that is, $2^{n-1} - m < m + 1$ and hence, $T_1 y \cup yT_1 = \langle x\rangle y$. Thus, $e(T) = 2$. □

We next show the existence of exhaustive subsets $T \subseteq Q_{2^n}$ with $e(T) = 3$.

Proposition 3.3. *Let $T = \{1, x, \ldots, x^m, y\} \subseteq Q_{2^n}$, $n \geq 3$, where $m = \left\lceil \frac{2^{n-1}}{3} \right\rceil$. Then $e(T) = 3$.*

Proof. Let $T_1 = \{1, x, x^2, \ldots, x^m\} \subseteq T$. Since $2m = 2\left\lceil \frac{2^{n-1}}{3} \right\rceil < 2\left\lceil \frac{2^{n-1}}{2} \right\rceil = 2^{n-1}$, so $T_1^2 = \{1, x, x^2, \ldots, x^{2m}\} \nsubseteq \langle x \rangle$.

We compute $T_1^2 y = \{y, xy, \ldots, x^{2m}y\}$ and $yT_1^2 = \{x^{2^{n-1}-2m}y, x^{2^{n-1}-2m+1}y, \ldots, x^{2^{n-1}-1}y, y\}$. Since $m = \left\lceil \frac{2^{n-1}}{3} \right\rceil > 2^{n-3}$, so $4m > 2^{n-1}$ and hence, $2m > 2^{n-1} - 2m$. This implies that $T_1^2 y \cup yT_1^2 = \{y, xy, \ldots, x^{2^{n-1}-1}y\}$. Therefore, $\langle x \rangle y \subseteq T^2$.

Note that $3m = 3\left\lceil \frac{2^{n-1}}{3} \right\rceil > 3\left(\frac{2^{n-1}}{3}\right) = 2^{n-1}$, so $\langle x \rangle = T_1^3 \subseteq T^3$ and $\langle x \rangle y \subseteq T^2 \subset T^3$. Hence, $T^3 = Q_{2^n}$ which implies that $e(T) = 3$. $\square$

By Proposition 2.1, we have the following:

Corollary 3.1. *Let $m \in \{\left\lceil \frac{2^{n-1}}{3} \right\rceil, \left\lceil \frac{2^{n-1}}{3} \right\rceil + 1, \ldots, 2^{n-2} - 1\}$ where $n \geq 3$. If $T = \{1, x, x^2, \ldots, x^m, y\} \subseteq Q_{2^n}$, then $e(T) \leq 3$.*

Proof. The result is obvious since the size of the subset T in this corollary is greater than or equal to the size of the subset T stated in Proposition 3.3. $\square$

Now note that if S is a non-exhaustive subset of G, then the set $\bar{S} = S \cup \{1\}$ is exhaustive. Indeed, since $1 \in \bar{S}$, we have that $\bar{S}^k \subset \bar{S}^{k+1}$ for all positive integer k. It follows that $|\bar{S}| < |\bar{S}^2| < \cdots < |\bar{S}^k| < |\bar{S}^{k+1}| < \cdots < |G|$; hence $|\bar{S}^n| = |G|$ for some positive integer n which implies that $\bar{S}$ is exhaustive. Thus, $T = \{1, x, y\}$ is exhaustive and to compute its exhaustion number, we make use of the following lemma which tells us which elements are in T^k.

Lemma 3.1. *Let $T = \{1, x, y\} \subseteq Q_{2^n}$, $n \geq 3$. Then*

(i) $\{1, x, x^2, \ldots, x^k\} \subseteq T^k \ \forall k \geq 2$.

(ii) $\{x^{2^{n-2}}, x^{2^{n-2}+1}, \ldots, x^{2^{n-2}+k-2}\} \subseteq T^k \ \forall k \geq 3$.

(iii) $\{y, xy, \ldots, x^{k-1}y\} \subseteq T^k \quad \forall k \geq 2$.

(iv) $\{x^{2^{n-2}-k+2}, x^{2^{n-2}-k+3}, \ldots, x^{2^{n-2}-1}\} \subseteq T^k \ \forall k \geq 3$.

(v) $\{x^{2^{n-2}}y, x^{2^{n-2}+1}y, \ldots, x^{2^{n-2}+k-3}y\} \subseteq T^k \ \forall k \geq 3$.

(vi) $\{x^{2^{n-1}-k+4}, x^{2^{n-1}-k+5}, \ldots, x^{2^{n-1}-1}\} \subseteq T^k \ \forall k \geq 5$.

(vii) $\{x^{2^{n-2}-k+3}y, x^{2^{n-2}-k+4}y, \ldots, x^{2^{n-2}+k-2}y\} \subseteq T^k \ \forall k \geq 4$.

(viii) $\{x^{2^{n-1}-k+1}y, x^{2^{n-1}-k+2}y, \ldots, x^{2^{n-1}-1}y\} \subseteq T^k \ \forall k \geq 3$.

Proof. (i) If $c \in T^i$, then $c \in T^j$ for all $j \geq i$ since $1 \in T$. Hence, for any positive integer $i \leq k$, we see that $x^i \in T^i \subseteq T^k$.

(ii) Note that $x^{2^{n-2}} = yy \in T^2 \subseteq T^i$ for $i \geq 2$ and $\{1, x, \ldots, x^{k-2}\} \subseteq T^{k-2}$.
Hence, $\{x^{2^{n-2}}, x^{2^{n-2}+1}, \ldots, x^{2^{n-2}+k-2}\} \subseteq T^2T^{k-2} = T^k$.

(iii) Since $y \in T$ and $\{1, x, \ldots, x^{k-1}\} \subseteq T^{k-1}$ (by (i)), we see that
$\{y, xy, \ldots, x^{k-1}y\} = \{1, x, \ldots, x^{k-1}\}y \subseteq T^{k-1}T = T^k$.

(iv) Since $\{x, x^2, \ldots, x^{k-2}\} \subseteq T^{k-2}$ and $y \in T$, it follows that
$\{xy, x^2y, \ldots, x^{k-2}y\} \subseteq T^{k-2}T = T^{k-1}$.
Then $\{x^{2^{n-2}-1}, x^{2^{n-2}-2}, \ldots, x^{2^{n-2}-k+2}\} = y\{xy, \ldots, x^{k-2}y\} \subseteq TT^{k-1} = T^k$.

(v) By (ii), we have that $\{x^{2^{n-2}}, x^{2^{n-2}+1}, \ldots, x^{2^{n-2}+k-3}\} \subseteq T^{k-1}$. Thus,
$\{x^{2^{n-2}}y, x^{2^{n-2}+1}y, \ldots, x^{2^{n-2}+k-3}y\} = \{x^{2^{n-2}}, x^{2^{n-2}+1}, \ldots, x^{2^{n-2}+k-3}\}y \subseteq T^{k-1}T = T^k$.

(vi) By selecting $\{x^{2^{n-2}+1}y, x^{2^{n-2}+2}y, \ldots, x^{2^{n-2}+k-4}y\} \subseteq T^{k-1}$ from (v), we have
$\{x^{2^{n-1}-1}, x^{2^{n-1}-2}, \ldots, x^{2^{n-1}-k+4}\} = y\{x^{2^{n-2}+1}y, x^{2^{n-2}+2}y, \ldots, x^{2^{n-2}+k-4}y\} \subseteq TT^{k-1} = T^k$.

(vii) The result is clear from (iv) and the fact that $y \in T$.

(viii) This is obvious since $\{x, x^2, \ldots, x^{k-1}\} \subseteq T^{k-1}$ (by (i)) and the relation $yx = x^{2^{n-1}-1}y$.

$\square$

We now determine the exhaustion number of $S = \{1, x, y\} \subseteq Q_{2^n}$, $n \geq 3$.

Proposition 3.4. *Let $S = \{1, x, y\} \subseteq Q_{2^n}$, $n \geq 3$. Then*

$$e(S) = \begin{cases} 3, & \text{if } n = 3, \\ 2^{n-3} + 3, & \text{if } n \geq 4. \end{cases}$$

Proof. It is straightforward to check that $e(S) = 3$ when $n = 3$ and $e(S) = 5$ when $n = 4$. Now suppose that $n \geq 5$.

By Lemma 3.1, parts (i), (iv), (ii) and (vi), we have that the sets $\{1, x, \ldots, x^{2^{n-3}+2}\}$, $\{x^{2^{n-3}}, x^{2^{n-3}+1}, \ldots, x^{2^{n-2}-1}\}$, $\{x^{2^{n-2}}, x^{2^{n-2}+1}, \ldots, x^{2^{n-2}+2^{n-3}}\}$ and $\{x^{2^{n-2}+2^{n-3}+2}, x^{2^{n-2}+2^{n-3}+3}, \ldots, x^{2^{n-1}-1}\}$, respectively, are all contained in $S^{2^{n-3}+2}$. However, $x^{2^{n-2}+2^{n-3}+1} \notin S^{2^{n-3}+2}$.

By Lemma 3.1, parts (i), (iv), (ii) and (vi) again, we have that the following sets are all contained in $S^{2^{n-3}+3}$: $\{1, x, \ldots, x^{2^{n-3}+3}\}$, $\{x^{2^{n-3}-1}, x^{2^{n-3}}, \ldots, x^{2^{n-2}-1}\}$, $\{x^{2^{n-2}}, x^{2^{n-2}+1}, \ldots, x^{2^{n-2}+2^{n-3}+1}\}$, $\{x^{2^{n-2}+2^{n-3}+1}, x^{2^{n-2}+2^{n-3}+2}, \ldots, x^{2^{n-1}-1}\}$. Therefore, $\langle x\rangle \subseteq S^{2^{n-3}+3}$ but $\langle x\rangle \nsubseteq S^{2^{n-3}+2}$.

Next, by parts (iii), (vii) and (viii) in Lemma 3.1, we have that $\{y, xy, \ldots, x^{2^{n-3}+1}y\}$, $\{x^{2^{n-3}+1}y, x^{2^{n-3}+2}y, \ldots, x^{2^{n-2}+2^{n-3}}y\}$ and $\{x^{2^{n-2}+2^{n-3}-1}y, x^{2^{n-2}+2^{n-3}}y, \ldots, x^{2^{n-1}-1}y\}$ are all contained in $S^{2^{n-3}+2}$. Hence, $\langle x\rangle y \subseteq S^{2^{n-3}+2}$.

By collecting the above results we have that $S^{2^{n-3}+2} \neq Q_{2^n}$ but $S^{2^{n-3}+3} = Q_{2^n}$. It follows that $e(S) = 2^{n-3} + 3$ for $n \geq 4$. $\square$

Corollary 3.2. *For any integer $k \in \{3, \ldots, 2^n\}$, there exists an exhaustive subset T of Q_{2^n}, $n \geq 3$, such that $|T| = k$.*

Proof. By Propositions 2.1 and 3.4, we see that for any subset $T \subseteq Q_{2^n}$ such that $T \supseteq \{1, x, y\} = S$, we have $e(T) \leq e(S) < \infty$. □

Another consequence of Proposition 3.4 is the following which gives upper and lower bounds for the exhaustion numbers of certain subsets of Q_{2^n}, $n \geq 3$.

Corollary 3.3. *Let* $T = \{1, x, \dots, x^m, y\} \subseteq Q_{2^n}$ *for* $m \in \{1, \dots, 2^{n-1} - 2\}$ *where* $n \geq 4$. *Then* $2 \leq e(T) \leq 2^{n-3} + 3$.

Proof. By Proposition 3.4, we see that $e(T) = 2^{n-3} + 3$ when $m = 1$. Next, by Proposition 3.2, we have $e(T) = 2$ when $m = 2^{n-1} - 2$. Hence, by Proposition 2.1, we conclude that $2 \leq e(T) \leq 2^{n-3} + 3$ for $m \in \{1, \dots, 2^{n-1} - 2\}$. □

Remark 3.2. Not all $i \in \{2, \dots, 2^{n-3} + 3\}$ as stated in Corollary 3.3 are realizable as the exhaustion number of a subset of the form $\{1, x, \dots, x^m, y\} \subseteq Q_{2^n}$ as shown below for the group Q_{32}.

Table 1: $e(T)$ where $T = \{1, x, \dots, x^m, y\}$ with size $|T| \in \{3, 4, \dots, 2^{n-1}\}$.

Q_{16} $(n = 4)$			Q_{32} $(n = 5)$		
m	$\lvert T\rvert$	$e(T)$	m	$\lvert T\rvert$	$e(T)$
1	3	5	1	3	7
2	4	4	2	4	5
3	5	3	3	5	5
4	6	2	4	6	4
5	7	2	$5, 6, 7$	$7, 8, 9$	3
6	8	2	$8, 9, \dots, 14$	$10, 11, \dots, 16$	2

The problem of finding the exhaustion numbers of subsets of finite non-abelian groups other than the dihedral and generalized quaternion groups remains open.

4 Conclusions

The exhaustion numbers of subsets of finite abelian groups has attracted the attention of researchers in the few past decades. This is due to its extensive applications of subsets of groups in the fields of number theory, coding, cryptography and etc. The study of exhaustion numbers of various subsets of abelian groups has been determined by Chin [4]. Factoring nonabelian groups into subsets is the new research direction and some related work has been completed, in particular, on the dihedral groups.

In this paper, we focus on the exhaustion numbers of the nonabelian groups, generalized quaternion groups Q_{2^n} for $n \geq 3$. Q_{2^n} has a unique algebraic structure for various applications. We identify the largest non-exhaustive subsets of Q_{2^n} of size 2^{n-1} and show that Q_{2^n} has no exhaustive subsets of size 2. In addition, the smallest positive integer k such that any subset $T \subseteq Q_{2^n}$ of size $\geq k$ is exhaustive is $2^{n-1} + 1$, with $e(T) = 2$. The exhaustion number of $S = \{1, x, y\} \subseteq Q_{2^n}$ is studied for $n \geq 3$, where size S is the smallest for a subset to be exhaustive. We also show that for any integer $k \in \{3, \dots, 2^n\}$, there exists an exhaustive subset T of Q_{2^n} such that $|T| = k$, and hence, there is no gaps in the size of an exhaustive subsets of Q_{2^n}. In summary, we determine the exhaustive and non-exhaustive subsets of generalized quaternion groups Q_{2n} for $n \geq 3$ and show that the generalized quaternion groups can be factored into subsets.

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Conflicts of Interest The authors declare no conflict of interest.

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